JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 33, No. 6, November–December 2010

Engineering Notes

Frequency-Independent Modal Damping for Flexural Structures via a Viscous "Geometric" Damping Model

George A. Lesieutre*
Pennsylvania State University,
University Park, Pennsylvania 16802

DOI: 10.2514/1.49864

Nomenclature

$a_m(t)$	=	modal coordinate
c_K, c_M, c_G	=	viscous damping coefficients for strain-based viscous damping, motion-based viscous damping, and rotation-based geometric viscous damping models
EI	=	flexural rigidity or stiffness of beam
L	=	length of beam
L_{el}	=	length of finite element
$[M]$, $[C]$, $[K]$	=	global mass, damping, and stiffness matrices
$M_v(x,t)$	=	viscous component of internal bending moment
m	=	mode number or index
$[m]$, $[c]$, $[k]$	=	elemental mass, damping, and stiffness matrices
$p_v(x,t)$	=	viscous component of external distributed lateral load acting on beam
$p_z(x,t)$	=	distributed lateral load acting on beam
$\{Q\}$	=	global load vector
$\{Q\}_{\text{elemental}}$	=	elemental load vector
$\{q\}$	=	vector of global finite element coordinates
$V_v(x,t)$	=	viscous component of internal shear force
$w(x,t)$	=	transverse displacement of beam neutral axis
w_a, w_b	=	transverse displacement at left and right sides of finite element
x	=	spatial coordinate that locates cross sections along the neutral axis of a beam
ζ_{Km}, ζ_{Mm}, ζ_{Gm}	=	modal damping ratios for strain-based viscous damping, motion-based viscous damping, and rotation-based geometric viscous damping models
ζ_m	=	modal damping ratio for mode m
$\zeta_{\text{norm}m}$	=	modal damping ratio for mode m, normalized by nominal damping ζ_{Gm}
θ_a, θ_b	=	rotation (slope) at left and right sides of finite element
ρA	=	mass per unit length of beam
ω_m	=	magnitude of natural frequency of vibration for mode m
ω_m^*	=	complex natural frequency of vibration for mode m

Presented as Paper 2010-3111 at the 51st AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference, Orlando, FL, 12–15 April 2010; received 14 March 2010; revision received 2 August 2010; accepted for publication 3 August 2010.

*Professor and Head, Department of Aerospace Engineering, 229 Hammond. Fellow AIAA.

I. Introduction

DAMPING must frequently be considered in the design of aerospace and other structures for a variety of reasons. These reasons include reducing dynamic response, reducing fatigue loads, ensuring aeroelastic stability, and providing adequate margin for active structural control in the presence of realistic uncertainty [1,2]. In all cases, the resulting system benefits from consideration of damping during the design process rather than as an afterthought [3].

To consider damping during the design process, accurate damping models are needed. The most popular damping models are linear and viscous in nature, in which case a generalized force proportional to and opposing a generalized velocity acts. In most cases, the main motivation for using viscous damping models is convenience: they provide dissipation in a time-domain model that can be used to obtain predictions of dynamic response to general forcing. Such viscous damping models, however, do not usually accurately reflect the physics underlying the actual dissipation mechanisms present. This shortcoming has motivated the development of more complex and accurate damping models: for instance, Golla–Hughes–McTavish [4] and anelastic displacement fields [5]. Such advanced damping models can capture the strong frequency dependence of the damping and stiffness exhibited by the high-loss viscoelastic materials that are frequently used in specialized damping treatments.

Experiments on typical built-up, lightly damped, aerospace structures, however, typically show relatively weak frequency-dependent damping [3]. The viscous damping models used to date suffer from the deficiency that predicted modal damping is strongly frequency dependent. Strain-based viscous damping, which corresponds to the case of stiffness-proportional damping, predicts modal damping that increases linearly with frequency (for flexural structures). Motion-based viscous damping, which corresponds to the case of mass-proportional damping, predicts modal damping that decreases linearly with frequency (for flexural structures). Complex modulus models can yield modal damping that is essentially independent of frequency, but the models are useful mainly in the frequency domain for determining the response to harmonic forcing [6].

There is a need for a viscous damping model that can yield frequency-independent modal damping. Such a model would be useful to designers and researchers who need a simple, time-domain damping model that exhibits realistic damping, damping that is, at most, weakly dependent on frequency. The main purpose of this Note is to develop such a model for flexural structures such as beams and plates.

II. Model Development and Analytical Results

To focus the development on a relatively simple example problem, consider the lateral motion of a planar beam, as shown in Fig. 1. The linear governing equation of motion, neglecting damping, is

$$\rho A(x)\frac{\partial^2 w}{\partial t^2}+\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 w}{\partial x^2}\right]=p_z(x,t) \tag{1}$$

For convenience, specialize to the case of a simply supported uniform beam, with the governing equation and boundary conditions as follows:

$$\rho A\frac{\partial^2 w}{\partial t^2}+EI\frac{\partial^4 w}{\partial x^4}=p_z(x,t) \tag{2}$$

$$w(0,t)=\frac{\partial^2 w(0,t)}{\partial x^2}=0 \tag{3}$$

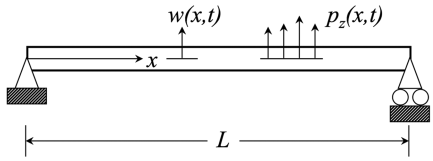

Fig. 1 A simply supported beam with distributed lateral force.

and

$$w(L,t) = \frac{\partial^2 w(L,t)}{\partial x^2} = 0$$

The solution of the associated undamped structural dynamics boundary-value eigenvalue problem involves mode shapes (eigenfunctions) having an integer number of half-sine waves:

$$w_m(x) = a_m \sin\left(\frac{m\pi x}{L}\right); \qquad m = 1, 2, \ldots \tag{4}$$

Note that a_m here represents a scaling factor; when time dependence is considered, it becomes a modal coordinate. The solution of the boundary-value eigenvalue problem also yields an expression for the natural frequencies:

$$\omega_m = m^2\pi^2\sqrt{\frac{EI}{\rho A L^4}} \tag{5}$$

As is well known for simply supported uniform beams, the nominal natural frequencies increase with the square of the mode number.

Next, consider several viscous damping models, which will be discussed as follows: the strain-based model (internal moment proportional to time rate of change of curvature) in Sec. II.A, the motion-based model (external distributed lateral force proportional to lateral velocity) in Sec. II.B, and the rotation-based model (internal shear force proportional to time rate of change of slope) in Sec. II.C.

None of these models are claimed to accurately capture the physics involved in damping: these viscous models may be said to provide energy dissipation with mathematical simplicity. In all cases, light damping is assumed, and all vibration modes of interest are underdamped; that is, $\zeta_m < 1$. The modal damping associated with each of these damping models is addressed in turn.

A. Strain-Based Viscous Damping

This model involves an internal moment that is proportional to the time rate of change of curvature; it could be said to be a dynamic (viscous) component of the bending moment. Fundamentally, it is associated with that part of the longitudinal stress that is proportional to the local strain rate. Expressions for this moment and the associated shear force follow:

$$M_v(x,t) = c_K \frac{\partial}{\partial t}\frac{\partial^2 w}{\partial x^2} \tag{6}$$

$$V_v(x,t) = \frac{\partial M_v}{\partial x} = \frac{\partial}{\partial x}\left(c_K \frac{\partial}{\partial t}\frac{\partial^2 w}{\partial x^2}\right) = c_K \frac{\partial}{\partial t}\frac{\partial^3 w}{\partial x^3} \tag{7}$$

Differentiating the shear force with respect to x adds an associated damping term to the equation of motion:

$$\rho A \frac{\partial^2 w}{\partial t^2} + \underbrace{c_K \frac{\partial}{\partial t}\frac{\partial^4 w}{\partial x^4}}_{\text{strain-based viscous damping}} + EI\frac{\partial^4 w}{\partial x^4} = p_z(x,t) \tag{8}$$

Assuming unforced motion in mode m, using Eq. (4), the following modal equation of motion is obtained:

$$\ddot{a}_m + \frac{c_K}{\rho A}\left(\frac{m\pi}{L}\right)^4 \dot{a}_m + \frac{1}{\rho A}\left[EI\left(\frac{m\pi}{L}\right)^4\right] a_m = 0 \tag{9}$$

And by comparing terms with those in the canonical unforced SDOF modal equation of motion,

$$\ddot{a}_m + 2\zeta_m\omega_m\dot{a}_m + \omega_m^2 a_m = 0 \tag{10}$$

one can develop an expression for the modal damping ratio ζ_m in terms of the beam properties. For this strain-based viscous damping model, the modal damping ratio for mode m is found as

$$\zeta_{Km} = \frac{c_K(m\pi/L)^2}{2(\rho A EI)^{1/2}} = \frac{c_K(m\pi/L)^2(EI/\rho A)^{1/2}}{2EI} = \frac{c_K\omega_m}{2EI} \tag{11}$$

In this case, the nominal modal damping increases with the square of the mode number: that is, with the nominal natural frequency. In practice, the viscous damping coefficient c_K may be chosen to provide the desired modal damping for a single mode; the modal damping of all other modes is dictated by Eq. (11).

B. Motion-Based Viscous Damping

This model involves an external distributed lateral force that is proportional to, and opposing, the transverse velocity:

$$p_v(x,t) = -c_M\frac{\partial w}{\partial t} \tag{12}$$

Moving this component of the external force to the left-hand side adds a term to the equation of motion, a term that is independent of the beam material considered:

$$\rho A\frac{\partial^2 w}{\partial t^2} + \underbrace{c_M\frac{\partial w}{\partial t}}_{\text{motion-based viscous damping}} + EI\frac{\partial^4 w}{\partial x^4} = p_z(x,t) \tag{13}$$

Assuming unforced motion in mode m, using Eq. (4), the following modal equation of motion is obtained:

$$\ddot{a}_m + \frac{c_M}{\rho A}\dot{a}_m + \frac{1}{\rho A}\left[EI\left(\frac{m\pi}{L}\right)^4\right]a_m = 0 \tag{14}$$

Again, one can develop an expression for the modal damping ratio ζ_m in terms of the beam properties by comparison to the canonical modal equation of motion (10). For this motion-based viscous damping model, the modal damping ratio for mode m is found as

$$\zeta_{Mm} = \frac{c_M}{2(\rho A EI)^{1/2}(m\pi/L)^2} = \frac{c_M(1/\rho A)}{2(EI/\rho A)^{1/2}(m\pi/L)^2} = \frac{c_M}{2\rho A\omega_m} \tag{15}$$

In this case, the nominal modal damping decreases with the square of the mode number: that is, with the nominal natural frequency. Again, in practice, the viscous damping coefficient c_M may be chosen to provide the desired modal damping for a single mode; the modal damping of all other modes is dictated by Eq. (15).

C. Rotation-Based Geometric Viscous Damping

This damping model involves an internal shear force that is proportional to the time rate of change of the slope; it could be said to be a dynamic (viscous) component of the shear force. This lateral force is analogous to that associated with a longitudinal (tensile) force carried by a deformed beam, in which the lateral (shear) component of the longitudinal force is proportional to the local slope:

$$V_v(x,t) = -c_G\frac{\partial}{\partial t}\frac{\partial w}{\partial x} \tag{16}$$

Alternatively, this model may be considered to involve an internal moment that is proportional to the transverse velocity:

$$M_v(x,t) = -c_G \frac{\partial w}{\partial t} \tag{17}$$

Differentiating the shear force with respect to x adds an associated damping term to the equation of motion. The sign of the term is such that it is dissipative:

$$\rho A \frac{\partial^2 w}{\partial t^2} - \underbrace{c_G \frac{\partial}{\partial t}\frac{\partial^2 w}{\partial x^2}}_{\text{geometric viscous damping}} + EI\frac{\partial^4 w}{\partial x^4} = p_z(x,t) \tag{18}$$

Assuming unforced motion in mode m, using Eq. (4), the following modal equation of motion is obtained:

$$\ddot{a}_m + \frac{c_G}{\rho A}\left(\frac{m\pi}{L}\right)^2 \dot{a}_m + \frac{1}{\rho A}\left[EI\left(\frac{m\pi}{L}\right)^4\right] a_m = 0 \tag{19}$$

Again, one can develop an expression for the modal damping ratio ζ_m in terms of the beam properties by comparison to the canonical modal equation of motion (10). For this rotation-based "geometric" viscous damping model, the modal damping ratio for mode m is found as

$$\zeta_{Gm} = \frac{c_G}{2\rho A \omega_m}\left(\frac{m\pi}{L}\right)^2 = \frac{c_G}{2(EI\rho A)^{1/2}} \tag{20}$$

In this case, the nominal modal damping is independent of the mode number and the nominal natural frequency. In practice, the viscous damping coefficient c_G may be chosen to provide the desired constant modal damping for all modes (assuming simply supported boundary conditions). Also note that the vibration modes, given by Eq. (4) for simply supported boundary conditions, are real.

III. Numerical (Finite Element) Results

The rotation-based geometric damping model was implemented in a finite element context. As shown in Fig. 2, a two-node element having four displacement degrees of freedom (a lateral displacement and slope at each end) was used, with the kinematic assumptions associated with the Bernoulli–Euler beam theory. Cubic interpolation functions are used to represent the lateral displacements of points on the neutral axis of the beam [7].

Equations (21a–21d) provide the associated elemental mass, stiffness, and damping matrices, as well as the elemental load vector [8], with the elemental degrees of freedom ordered as $\lfloor w_a \quad \theta_a \quad w_b \quad \theta_b \rfloor$:

$$[m] = \frac{\rho A L_{el}}{420}\begin{bmatrix} 156 & 22L_{el} & 54 & -13L_{el} \\ 22L_{el} & 4L_{el}^2 & 13L_{el} & -3L_{el}^2 \\ 54 & 13L_{el} & 156 & -22L_{el} \\ -13L_{el} & -3L_{el}^2 & -22L_{el} & 4L_{el}^2 \end{bmatrix} \tag{21a}$$

$$[c] = \frac{c_G}{30L_{el}}\begin{bmatrix} 36 & 3L_{el} & -36 & 3L_{el} \\ 3L_{el} & 4L_{el}^2 & -3L_{el} & -L_{el}^2 \\ -36 & -3L_{el} & 36 & -3L_{el} \\ 3L_{el} & -L_{el}^2 & -3L_{el} & 4L_{el}^2 \end{bmatrix} \tag{21b}$$

$$[k] = \frac{EI}{L_{el}^3}\begin{bmatrix} 12 & 6L_{el} & -12 & 6L_{el} \\ 6L_{el} & 4L_{el}^2 & -6L_{el} & 2L_{el}^2 \\ -12 & -6L_{el} & 12 & -6L_{el} \\ 6L_{el} & 2L_{el}^2 & -6L_{el} & 4L_{el}^2 \end{bmatrix} \tag{21c}$$

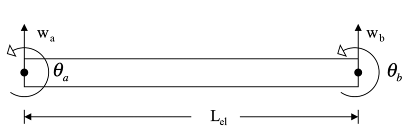

Fig. 2 Two-node planar beam finite element.

$$\{Q\}_{\text{elemental}} = p_z L_{el}\begin{Bmatrix} 1/2 \\ +L_{el}/12 \\ 1/2 \\ +L_{el}/12 \end{Bmatrix} \tag{21d}$$

Note that the elemental damping matrix [Eq. (21b)] is identical in structure to the geometric stiffness matrix encountered in structural stability problems [9]. Furthermore, note that this damping model is not proportional, in that the damping matrix is not a linear combination of the mass and stiffness matrices.

In the usual way [10], assembling the global mass, stiffness, and damping matrices from the elemental matrices, the global force vector from elemental load vectors and global point forces, then imposing the constraints associated with the kinematic boundary conditions yields matrix equations of motion:

$$[M]\{\ddot{q}\} + [C]\{\dot{q}\} + [K]\{q\} = \{Q\} \tag{22}$$

The associated structural dynamics eigenvalue problem may be posed (nonuniquely) in first-order form as

$$\omega_m^* \begin{bmatrix} [M] & [0] \\ [0] & [I] \end{bmatrix}\begin{Bmatrix} \{\dot{q}\} \\ \{q\} \end{Bmatrix} = -\begin{bmatrix} [C] & [K] \\ -[I] & [0] \end{bmatrix}\begin{Bmatrix} \{\dot{q}\} \\ \{q\} \end{Bmatrix} \tag{23}$$

where ω_m^* is understood to be complex and of the form,

$$\omega_m^* = -\zeta_m \omega_m \pm i\omega_m\sqrt{1-\zeta_m^2} \tag{24}$$

A nominal case in which 50 elements are used to model a uniform, simply supported, planar beam is discussed herein. The eigenvalue problem [Eq. (23)] was solved numerically using nominal parameter values, and normalized results are presented. For 50 elements, perhaps only the first 25 finite-element-based modes capture, with reasonable accuracy, the mode frequencies and eigenvectors associated with the continuous boundary-value eigenvalue problem. The modal damping ratios were calculated from the complex eigenvalues as follows:

$$\zeta_m = -Re(\omega_m^*)/|\omega_m^*| \tag{25}$$

and normalized to the (constant) value predicted by Eq. (20).

$$\zeta_{\text{norm}m} = \frac{\zeta_m}{\zeta_{Gm}} = \frac{\zeta_m}{c_G/2(EI\rho A)^{1/2}} \tag{26}$$

Figure 3 shows the normalized modal damping ratio $\zeta_{\text{norm}m}$ versus mode number m for the first 50 modes of the discretized model. The modal damping for the first 25 modes is very nearly constant, varying by less than 0.5% from the value associated with the first mode. Over the first 49 modes, the variation is less than 2.5%. Furthermore, the numerical value of the damping of the first mode agrees very well with that predicted by Eq. (20).

For completeness, Fig. 4 shows the normalized modal damping ratio $\zeta_{\text{norm}m}$ versus mode number m for all the modes of the discretized model. As can be seen, the modal damping for all the modes varies by less than 20% from the low-mode-number damping, even though the higher modes are increasingly inaccurate, with little physical significance. This property might also make the subject damping model useful in flexural wave propagation problems.

The nature of the mode shapes is also of interest, as it is somewhat more convenient to use real modes in structural dynamics analysis. As noted at the end of Sec. II, the vibration modes for simply supported boundary conditions [Eq. (4)] are real. This is perhaps unexpected, as this rotation-based geometric viscous damping model is not a proportional damping model (a special kind of viscous damping model that is known to yield real vibration modes [11]).

In general, the eigenvectors of a discretized system described by Eqs. (22) and (23) are complex. If the displacement components $\{q\}$ of the eigenvectors obtained using the present damping model are normalized with respect to the mass matrix {$[M]$ in Eq. (22)}, the relative magnitudes of the real and imaginary parts can be compared.

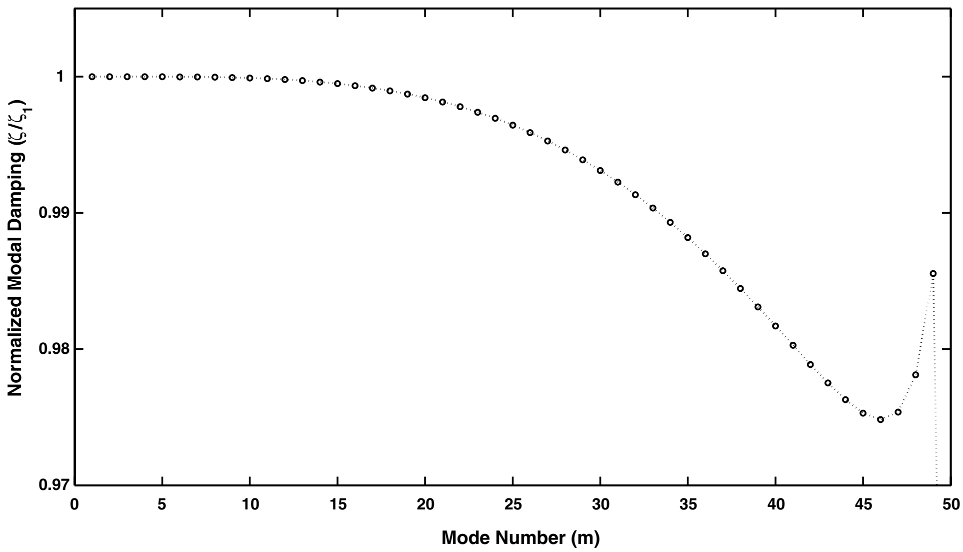

Fig. 3 Normalized modal damping vs mode number (lowest modes).

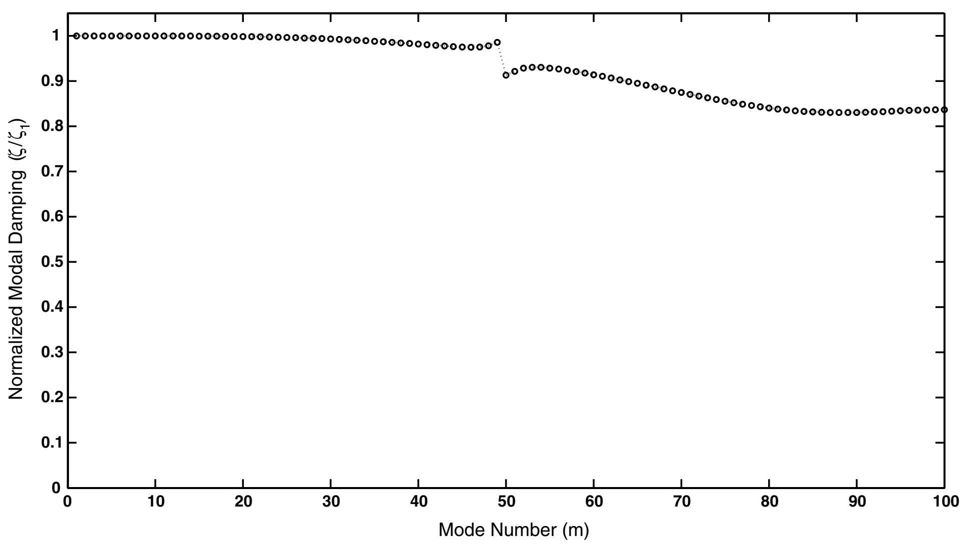

Fig. 4 Normalized modal damping vs mode number (all modes).

Based on numerical results for nominal modal damping of 5%, the lateral motion of points on the neutral axis of the beam vibrating in mode 1 is in phase to about one part in 10^9. For mode 25, this increases to about one part in 10^5. For practical purposes, these correspond to real mode shapes, consistent with Eqs. (4) and (20).

IV. Conclusions

Viscous damping models available to date suffer from the deficiency that predicted modal damping is strongly frequency dependent, a situation not typically encountered in experiments with built-up structures lacking specialized damping treatments. For flexural structures, strain-based viscous damping, which corresponds to the case of stiffness-proportional damping, yields modal damping that increases linearly with frequency. Motion-based viscous damping, which corresponds to the case of mass-proportional damping, yields modal damping that decreases linearly with frequency.

The proposed rotation-based model introduces a viscous geometric damping term in which an internal shear force is proportional to the time rate of change of the slope. This lateral force is analogous to that associated with a longitudinal force carried by a deformed beam, a situation in which the lateral (shear) component of the longitudinal force is proportional to the local slope. In a discretized (finite element) context, the resulting damping matrix closely resembles the geometric stiffness matrix used to account for the effects of membrane loads on lateral stiffness.

For uniform simply supported flexural beams, this model yields: 1) modal damping that is constant and independent of frequency, as well as 2) real vibration modes. Numerical finite element analysis confirms these analytical conclusions. The fact that the vibration modes are real is perhaps surprising, as the damping matrix is not a linear combination of the stiffness and mass matrices. Such a viscous damping model should prove useful to researchers and engineers who need a time-domain damping model that exhibits frequency-independent damping in flexural structures.

An interesting question is whether this relationship holds for different boundary conditions. Because the relationship between modal frequencies and mode number is sensitive to boundary conditions, especially for low mode numbers, one might expect that the associated modal damping would deviate from uniformity. Continuing research should investigate this question, as well as the extension of this model to more complex structural members, such as plates.

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